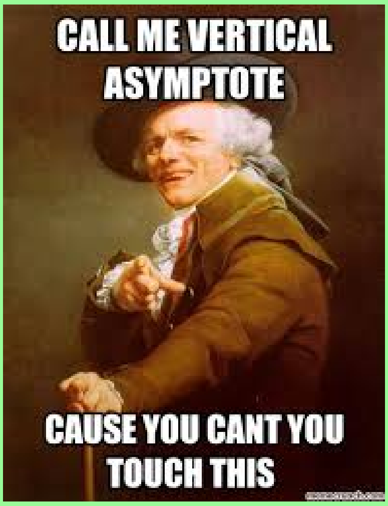
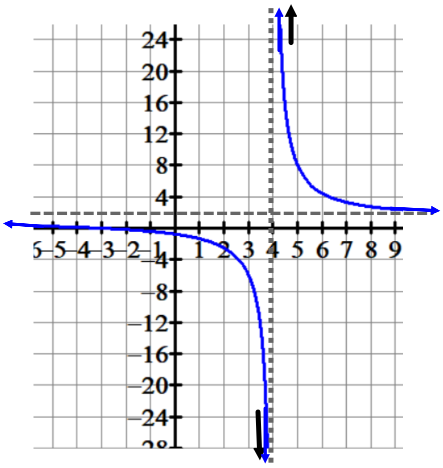
Lesson 2
Horizontal Asymptotes of Rational Functions and Behaviour (at VA and end)

Recall



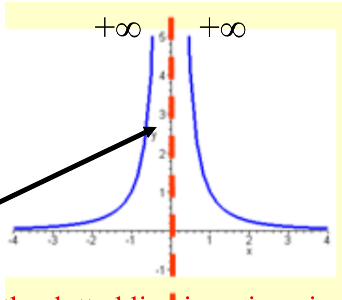
When sketching rational functions you must check the behaviour at the vertical asymptote!!



for this example:
on the left side of the VA the graph goes on to -∞

on the right side of the VA the graph goes on to +∞

The Graph of $f(x) = \dfrac{1}{x^2}$ look like this



the vertical asymptote is at x=0 because the denominator cannot be 0. As the graph approached) it gets closer and closer but will never cross.
How do we know the direction it goes as it approaches 0? **This is where we must look at the behaviour at the VA**

the dotted line is an imaginary line that the graph cannot cross

Behaviour?

choose two points that are close to the VA on either side of the VA (you could say we are straddling the VA)

VA
x=0

$\frac{1}{x^2}$	**-0.1**	**0.1**
	$\frac{+}{+}$	$\frac{+}{+}$
	$+\infty$	$+\infty$

$$f(x) = \frac{(x+3)(x-5)}{(x+4)(x-1)}$$

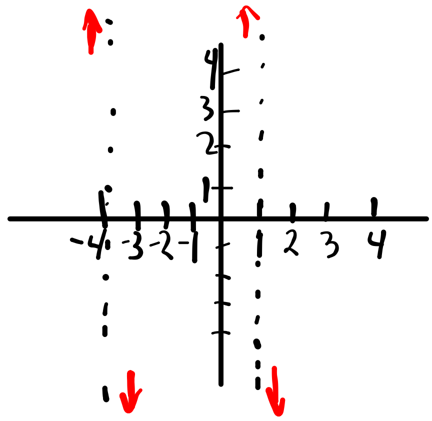
VA's: $x=-4$, $x=1$

$x=-4$

-4.1	-3.9
$\frac{- \ -}{- \ -}$	$\frac{- \ -}{+ \ -}$
$+\infty$	$-\infty$

$x=1$

0.9	1.1
$\frac{+ \ -}{+ \ -}$	$\frac{+ \ -}{+ \ +}$
$+\infty$	$-\infty$



Lets look at $f(x) = \dfrac{2x+12}{x-3}$ $\xrightarrow{\text{factor}}$ $f(x) = \dfrac{2(x+6)}{x-3}$

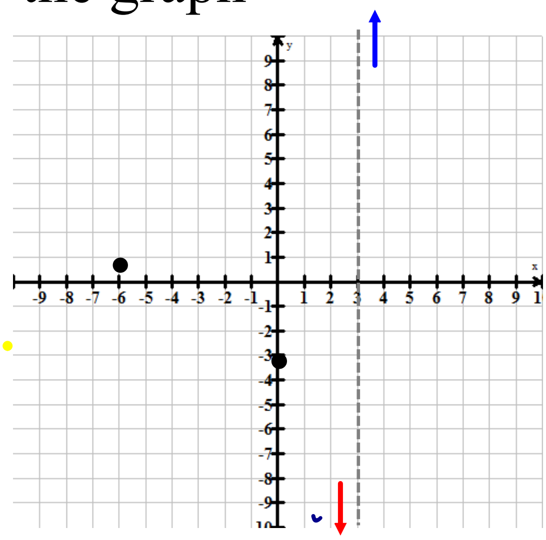
Determine

x-int (zero of numerator) = -6

y-int (let x=0 and solve for y) = -4

VA (undefines values of x, aka zeros of denominator) x=3

all this to get the graph



Now check out the behaviour at the VA

x=3

$f(x) = \dfrac{2(x+6)}{x-3}$

2.9	3.1
+ +	+ +
-	+
= $-\infty$	= $+\infty$

look for the sign (+ or -) of each factor that way you ae dealing with only signs.

the rest of the graph comes soon

Lets look at $f(x) = \dfrac{x^2-8x+12}{x^2-9}$ $\xrightarrow{\text{factor}}$ $f(x) = \dfrac{(x-2)(x-6)}{(x-3)(x+3)}$

Determine

x-int : 2 , 6

y-int : -12/9

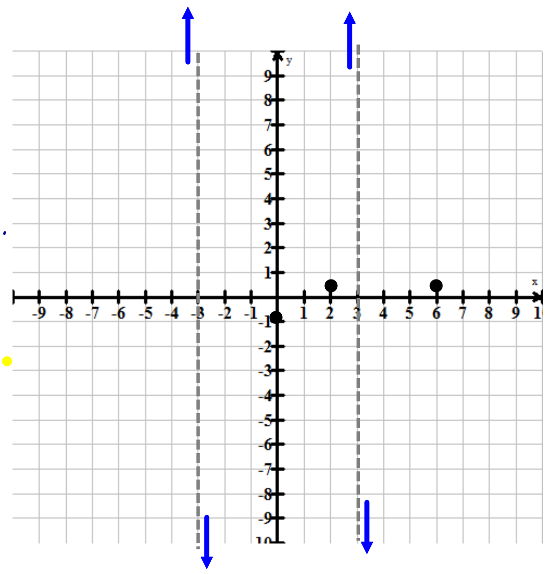
VA: x=-3 , x=3

all this to get the graph

Now check out the behaviour at the VA's

$f(x) = \dfrac{(x-2)(x-6)}{(x-3)(x+3)}$

x= -3			x= 3	
-3.1	-2.9		2.9	3.1
- -	- -		+ -	+ -
- -	- +		- +	+ +
+∞	-∞		+∞	-∞



the rest of the graph comes soon

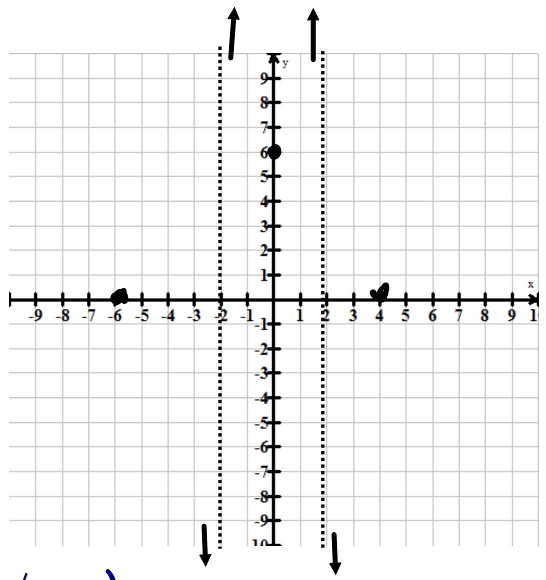
$$f(x) = \frac{(x+6)(x-4)}{(x+2)(x-2)} \qquad f(x) = \frac{x^2+2x-24}{x^2-4}$$

x -int: -6 and 4

y-int: 6

VA: x=2 , x=-2

HA: y=1



$$f(x) = \frac{(x+6)(x-4)}{(x+2)(x-2)}$$

x= -2		x= 2	
-2.1	-1.9	1.9	2.1
+ -	+ -	+ -	+ -
- -	+ -	+ -	+ +
-∞	+∞	+∞	-∞

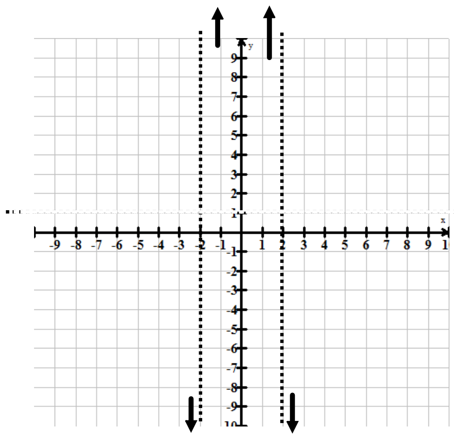
$$\lim_{x \to 2} \frac{(x+6)(x-4)}{(x+2)(x-2)} \Rightarrow \pm\infty$$

$$y = \frac{(x+6)(x-4)}{(x+2)(x-2)} \; ; \; y = \frac{x^2+2x-24}{x^2-4}$$

x-int -6 and 4
(zeros of num)

y-int: 6
(let $x=0$)

VA's $x=2, x=-2$
(zeros of den)



behaviour at VA

x = -2		x = 2	
-2.1	-1.9	1.9	2.1
$+ \; -$	$+ \; -$	$+ \; -$	$+ \; -$
$-$	$+ \; -$	$+ \; -$	$+ \; +$
$-\infty$	$+\infty$	$+\infty$	$-\infty$

Horizontal Asymptotes

$$\lim_{x\to\infty}\frac{3x^2}{x^2+1}$$

" limit as x approaches ∞

this means what happens to the function $f(x) = \dfrac{3x^2}{x^2+1}$ as x approaches infinity (as x gets infinitely large)

$3 = \dfrac{3x^2}{x^2+1}$

$3(x^2+1) = 3x^2$

$\cancel{3x^2} + 3 = \cancel{3x^2}$

$3 = 0$ NO

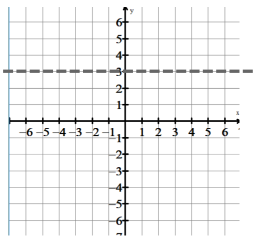
x	10	100	1,000	10,000	100,000
f(x)	2.97	2.9997	3	3	3

sub x values into the function, let x get very large....function approaches 3

Thus:

$$\lim_{x\to\infty}\frac{3x^2}{x^2+1} = 3$$

this means there is a horizontal asymptote at y=3



$\frac{5}{x}$ if x is very very large
will approach 0

$\frac{\#}{x, x^2, x^3 \ldots}$ approach 0.

$\div$ top & bottom by x^2

$\frac{3x^2}{x^2+1}$

$\frac{\frac{3\cancel{x^2}}{\cancel{x^2}}}{\frac{\cancel{x^2}}{\cancel{x^2}} + \frac{1}{x^2}} \rightarrow \frac{3}{1+\frac{1}{x^2}}$

$\frac{3}{1+0} = 3$

Strategy

$$\lim_{x\to\infty}\frac{3x^2}{x^2+1}$$

$$\lim_{x\to\infty}\frac{\frac{3\cancel{x^2}}{\cancel{x^2}}}{\frac{\cancel{x^2}}{\cancel{x^2}}+\frac{1}{x^2}}$$

$$\lim_{x\to\infty}\frac{3}{1+\frac{1}{x^2}}$$

$$=\frac{3}{1+0}$$

$$=3$$

- If r is a positive rational number and c is a number then

$$\lim_{x\to\infty}\frac{c}{x^r}=0$$ *this is because a constant divided by a very large number is 0*

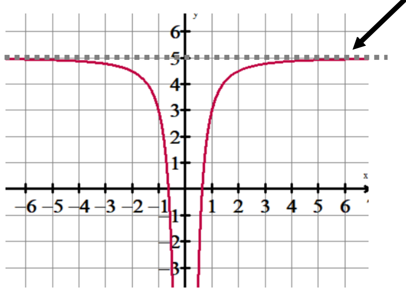
using this idea we can divide both numerator and denominator by the highest power of x, that is x^2. THat does not change teh value of teh expression

$$\lim_{x\to\infty}\left(5-\frac{2}{x^2}\right) = 5$$

$$\lim_{x\to\infty} 5 - \lim_{x\to\infty} \cancel{\frac{2}{x^2}} \to 0$$

$$5 - 0 = 5$$

because a constant divided by a very large number is 0



thus there is a line at y=5 that the function approaches as x approaches +∞ or -∞

$$\lim_{x\to\infty}\frac{2x-1}{x+1}$$ ←

divide numerator and denominator by the highest power of x (that is x)

$$\frac{\frac{2\cancel{x}}{\cancel{x}}-\frac{1}{x}}{\frac{\cancel{x}}{\cancel{x}}+\frac{1}{x}}$$

$$\lim_{x\to\infty}\frac{2-1/x}{1+1/x}$$

$$\frac{2-0}{1+0}=2 \quad \text{HA}$$

$$f(x)=\frac{5x^3-6x^2+7}{2x^3-9}$$

$$\frac{\frac{5\cancel{x^3}}{\cancel{x^3}}-\frac{6x^2}{x^3}+\frac{7}{x^3}}{\frac{2x^3}{x^3}-\frac{9}{x^3}}$$

$$\frac{5-\frac{6}{x}+\frac{7}{x^3}}{2-9/x^3}$$

$$\frac{5-0+0}{2-0}=5/2$$

Example 1

divide by x^3

$$\lim_{x\to\infty} \frac{5x^3-6x+7}{12-2x-7x^3}$$

$$\lim_{x\to\infty} \frac{\frac{5x^3}{x^3}-\frac{6x}{x^3}+\frac{7}{x^3}}{\frac{12}{x^3}-\frac{2x}{x^3}-\frac{7x^3}{x^3}}$$

$$\lim_{x\to\infty} \frac{5-\frac{6}{x^2}+\frac{7}{x^3}}{\frac{12}{x^3}-\frac{2}{x^2}-7}$$

$$\lim_{x\to\infty} \frac{5-0+0}{0-0-7}$$

$= -5/7$

Example 2

divide by x^5

$$\lim_{x\to\infty} \frac{2x^5-6x^2}{4x^4-2x-2}$$

$$\lim_{x\to\infty} \frac{\frac{2x^5}{x^5}-\frac{6x^2}{x^5}}{\frac{4x^4}{x^5}-\frac{2x}{x^5}-\frac{2}{x^5}}$$

$$\lim_{x\to\infty} \frac{2-\frac{6}{x^3}}{\frac{4}{x}-\frac{2}{x^4}-\frac{2}{x^5}}$$

$$\lim_{x\to\infty} \frac{2-0}{0-0-0}$$

$= \frac{2}{0}$

undefined

There is a shortcut for rational functions, (so you do not have to divide by the highest power each time)
-choose the highest power of x:
-use the coefficients of the highest power. (just make sure each polynomial is expanded)

highest power 3

Example 1 $\lim_{x\to\infty} \frac{5x^3-6x+7}{12-2x-7x^3} = \frac{5}{-7}$

highest power 5

Example 2 $\lim_{x\to\infty} \frac{2x^5-6x^2}{4x^4-2x-2} = \frac{2}{0}$ **undefined**

Shortcuts: for determining limits of infinity for rational functions!!!! using the coefficients of the highest power. $\boxed{\frac{f(x)}{g(x)}}$

- 1. If the numerator's exponent is greater than the denominator then the limit is infinity or negative infinity
- 2. If the numerator's exponent is less than the denominator then the limit is zero
- 3. If the numerator's exponent is equal to the denominator then the limit is the ratio of coefficients

Examples

$$\lim_{x\to\infty}\frac{3x^2}{x^2+1} = 3$$

$$\lim_{x\to-\infty}\frac{3x^2+7}{x^5+1} = 0$$

$0/1 =$

$$\lim_{x\to-\infty}\frac{5x^2+8x^3}{4x^2+1} \quad \frac{8}{0}$$

undefined

$$\lim_{x\to\infty}\frac{(8x+7)(5x+2)}{6x^2+1}$$

expand: $\frac{40x^2+51x+14}{6x^2+1}$

$= 40/6$

$20/3$

- 1. If the numerator's exponent is greater than the denominator then the limit is infinity or negative infinity

ex: $\lim_{x \to \infty} \frac{2x^3+5}{3x^2+1} = \frac{2}{0} = \infty$ (or DNE)

does not exist

ex: $\lim_{n\to\infty} \frac{8(n)(n+1)(2n+1)}{6n^2}$

expand before finding the highest power

$= \lim_{n\to\infty} \frac{8n^3+12n^2+4n}{3n^2}$ =DNE

$= \frac{8}{0}$

- 2. If the numerator's exponent is less than the denominator then the limit is zero

ex: $\lim_{x \to \infty} \frac{2x+5}{3x^2+1} = \frac{0}{3} = 0$

$\lim_{x\to\infty} \frac{9x^2-3x+11}{x^3-2x^2-7} = \frac{0}{1} = 0$

- 3. If the numerator's exponent is equal to the denominator then the limit is the ratio of coefficients

ex: $\lim_{x \to \infty} \frac{2x-1}{x+1} = \frac{2}{1} = 2$

ex: $\lim_{x\to\infty} \frac{12x^4-5x^3+0x^2-1}{16-2x^4} = \frac{12}{-2} = -6$

Horizontal Asymptotes for Rational functions are the limits of infinity

Determine the horizontal asymptotes of the functions $y =$

a. $f(x) = \dfrac{8x^2 - 6x - 9}{16x^2 - 9}$

$y = 8/16 = 1/2.$

b. $f(x) = \dfrac{x^2 - 2x - 48}{x^2 + 4x - 5}$

c. $f(x) = \dfrac{5x^2 - 10x - 240}{2x^2 - 7x + 6}$

d. $f(x) = \dfrac{x^4 - 7x^3 - 30x^2}{2x^2 + 3x - 2}$

e. $f(x) = \dfrac{10x^3 - 80}{x^3 + 8}$

f. $f(x) = \dfrac{12x - 60}{x^2 - 7x - 60}$

g. $f(x) = \dfrac{3x^2 + 14x + 5}{x^2 + 5}$

h. $f(x) = \dfrac{x^3 - 5x^2 + 2x + 8}{3x^4 - 81x}$

i. $f(x) = \dfrac{3x^4 - 2x^2 - 1}{x^4 - 16}$

Horizontal Asymptotes for Rational functions are the limits of infinity

Determine the horizontal asymptotes of the functions

a. $f(x)=\dfrac{8x^2-6x-9}{16x^2-9}$

HA: $y=\frac{8}{16}=\frac{1}{2}$

b. $f(x)=\dfrac{x^2-2x-48}{x^2+4x-5}$

HA: $y=\frac{1}{1}=1$

c. $f(x)=\dfrac{5x^2-10x-240}{2x^2-7x+6}$

HA: $y=\frac{5}{2}$

d. $f(x)=\dfrac{x^4-7x^3-30x^2}{2x^2+3x-2}$

HA: $y=\frac{1}{0}$ =undefined
no HA

e. $f(x)=\dfrac{10x^3-80}{x^3+8}$

HA: $y=\frac{10}{1}=10$

f. $f(x)=\dfrac{12x-60}{x^2-7x-60}$

HA: $y=\frac{0}{1}=0$

g. $f(x)=\dfrac{3x^2+14x+5}{x^2+5}$

HA: $y=\frac{3}{1}=3$

h. $f(x)=\dfrac{x^3-5x^2+2x+8}{3x^4-81x}$

HA: $y=\frac{0}{3}=0$

i. $f(x)=\dfrac{3x^4-2x^2-1}{x^4-16}$

HA: $y=\frac{3}{1}=3$

a) $f(x)=\dfrac{8x^2-6x-9}{16x^2-9}$

$f(x)=\dfrac{(4x+3)(2x-3)}{(4x-3)(4x+3)}$

x-int: $-3/4$, $3/2$

y-int $-9/-9=1$

VA: $x=3/4$, $-3/4$

HA $y=8/16=1/2$

HA, VA, Intercepts.doc

Determine the intercepts, horizontal and vertical asymptotes

x-int $-\frac{3}{4}, \frac{3}{2}$
y-int $\frac{-9}{-9} = 1$

a. $f(x) = \dfrac{8x^2 - 6x - 9}{16x^2 - 9}$, $f(x) = \dfrac{(4x+3)(2x-3)}{(4x-3)(4x+3)}$

b. $f(x) = \dfrac{x^2 - 2x - 48}{x^2 + 4x - 5}$, $f(x) = \dfrac{(x-8)(x+6)}{(x+5)(x-1)}$

$x^2 - 2x - 45$

c. $f(x) = \dfrac{5x^2 - 10x - 240}{2x^2 - 7x + 6}$ $f(x) = 5($

d. $f(x) = \dfrac{x^4 - 7x^3 - 30x^2}{2x^2 + 3x - 2}$ $f(x) = \dfrac{x^2(x-10)(x+3)}{(2x-1)(x+2)}$

e. $f(x) = \dfrac{10x^3 - 80}{x^3 + 8}$ $f(x) = \dfrac{10(x-2)(x^2+2x+4)}{(x+2)(x^2-2x+4)}$

f. $f(x) = \dfrac{12x - 60}{x^2 - 7x - 60}$ $f(x) = \dfrac{12(x-5)}{(x-12)(x+5)}$

g. $f(x) = \dfrac{3x^2 + 14x + 5}{x^2 + 5}$ → cannot factor

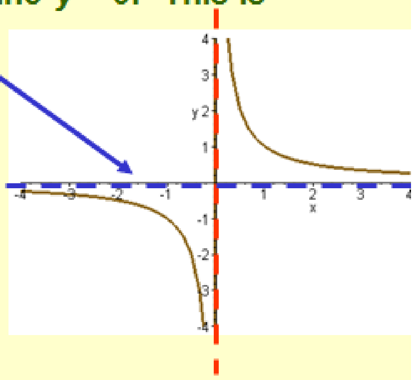
cannot factor

A graph will NEVER cross a vertical asymptote because the *x* value is “illegal” (would make the denominator 0)

A graph may cross a horizontal asymptote near the middle of the graph but will approach it when you move to the far right or left

Does the function $f(x)=\frac{1}{x}$ have an *x* intercept? $0 \neq \frac{1}{x}$

There is NOT a value that you can plug in for *x* that would make the function = 0. The graph approaches but never crosses the horizontal line *y* = 0. This is called a horizontal asymptote.



A graph will NEVER cross a vertical asymptote because the *x* value is "illegal" (would make the denominator 0)

A graph may cross a horizontal asymptote near the middle of the graph but will approach it when you move to the far right or left

RECAP: Graphing Rational Functions!!!!

1. x-intercepts: zeros of the numerator
   y-intercepts: let x=0 and solve for y

2. VA: zeros of the denominator
   HA- (limits of infinity)divide expanded forms by the highest power

3. Behaviour at the vertical asymptotes

all of this will help graph the rational function

graph will NEVER cross a VA but may cross a HA

Note: you can check to see if a function crosses a horizontal asymptote by making the y value equal to the HA and solving

$$f(x) = \frac{x^2-11x+18}{2x^2-9x+7}$$ factor if possible $$f(x) = \frac{(x-9)(x-2)}{(2x-7)(x-1)}$$

x-int: 9 and 2
y-int: 18/7
VA: x=7/2 , x=1
HA: y=1/2

Behaviour $\frac{(x-9)(x-2)}{(2x-7)(x-1)}$

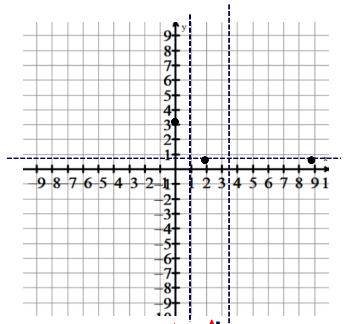
x=1

0.9	1.1
- -	- -
- -	- +
$+\infty$	$-\infty$

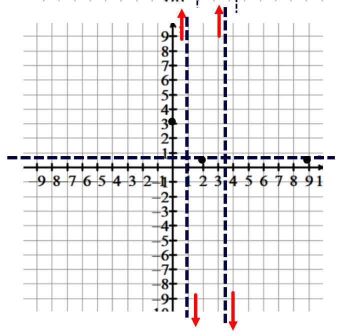
x=3.5

3.4	3.6
- +	- +
- +	+ +
$+\infty$	$-\infty$

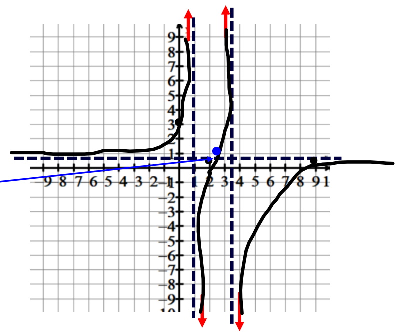
To Graph
1) Place intercepts, VA and HA with dashed lines



2) Place arrows to show behaviour



3) Sketch

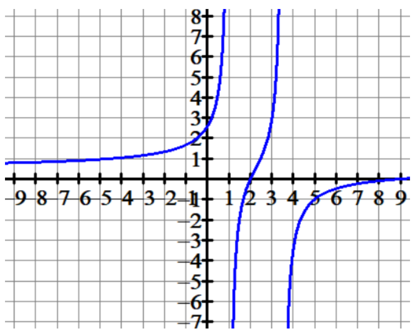


This is an example where the graph crosses the horizontal asymptote: if you want to find the exact point at which it crosses set: f(x) = HA

$$f(x) = 1/2 \quad \frac{x^2-11x+18}{2x^2-9x+7} = \frac{1}{2}$$

$$2(x^2-11x+18) = 2x^2-9x+7$$
$$\cancel{2x^2}-22x+36 = \cancel{2x^2}-9x+7$$
$$-22x-9x = 7-36$$
$$-31x = -29$$
$$x = -29/-31 \cong 0.94$$
$$(29/31 , 1.2)$$

Computer generated



sketch: $y = \dfrac{x^2-8x-50}{25-x^2}$; $y = \dfrac{x^2-8x-50}{(5-x)(5+x)}$ ← use QF OR $y = \dfrac{x^2-8x-50}{-1(x-5)(x+5)}$

x-int : $\dfrac{8 \pm \sqrt{264}}{2} < \begin{matrix} 12.1 \\ -4.1 \end{matrix}$

y-int: $-50/25 = -2$

VA: $x = 5,\ x = -5$

HA: $y = -1$

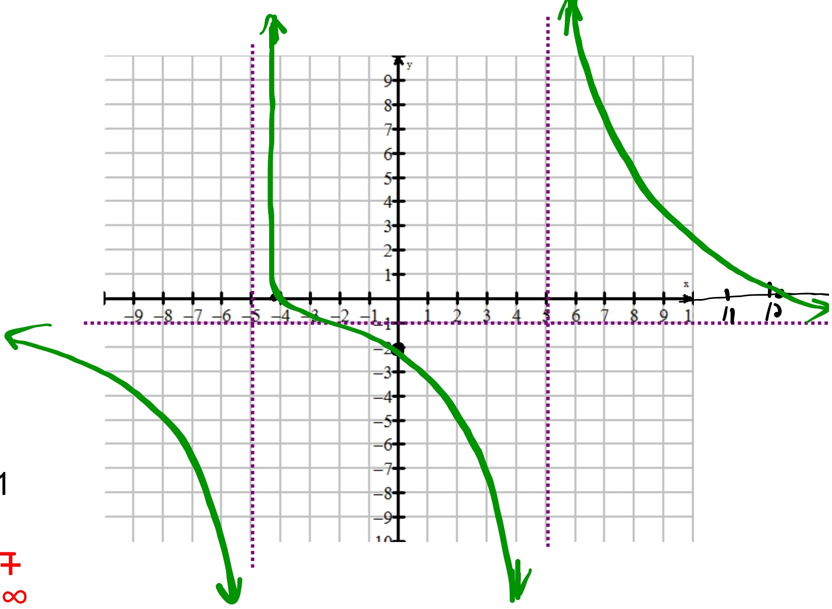
Behav:

-5.1	-4.9
$\dfrac{+}{+\ -}$	$\dfrac{+}{+\ +}$
$-\infty$	$+\infty$

x= -5

4.9	5.1
$\dfrac{-}{+\ +}$	$\dfrac{-}{-\ +}$
$-\infty$	$+\infty$

x= 5



$$f(x) = \frac{(2x-3)(x+4)}{(x+2)(x-3)} \quad ; \quad H(x) = \frac{2x^2+5x-12}{x^2-x-6}$$

x-int: $\frac{3}{2}$, -4

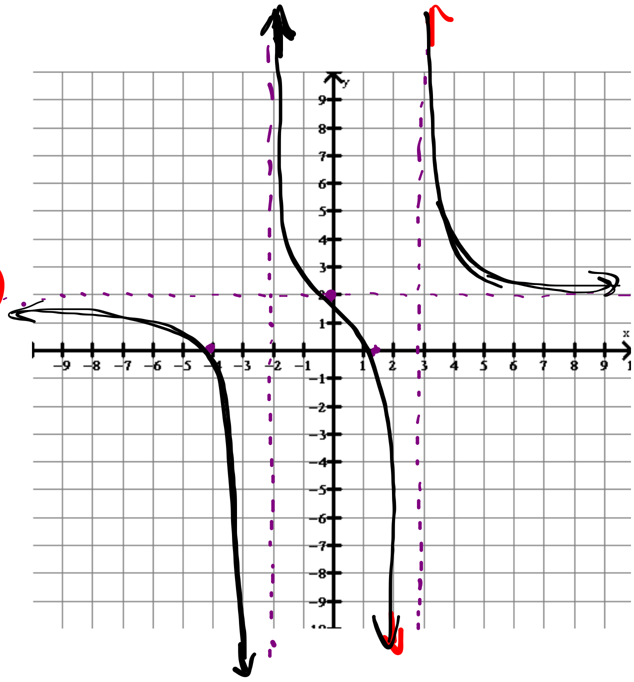
y-int: $\frac{-12}{-6} = 2$

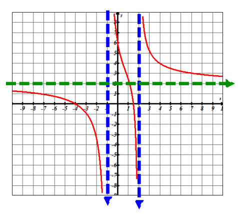
VA: $x = -2$, $x = 3$

HA: $y = 2$

Beh. VA: $x = -2$, $x = 3$

-2.1	-1.9	2.9	3.1
− +	− +	+ +	+ +
− −	+ −	+ −	+ +
$-\infty$	$+\infty$	$-\infty$	$+\infty$





$$y = \frac{(2x-1)(x+3)}{(x+6)^2}; \quad y = \frac{2x^2+5x-3}{x^2+12x+36}$$

x-int : $\frac{1}{2}$; -3

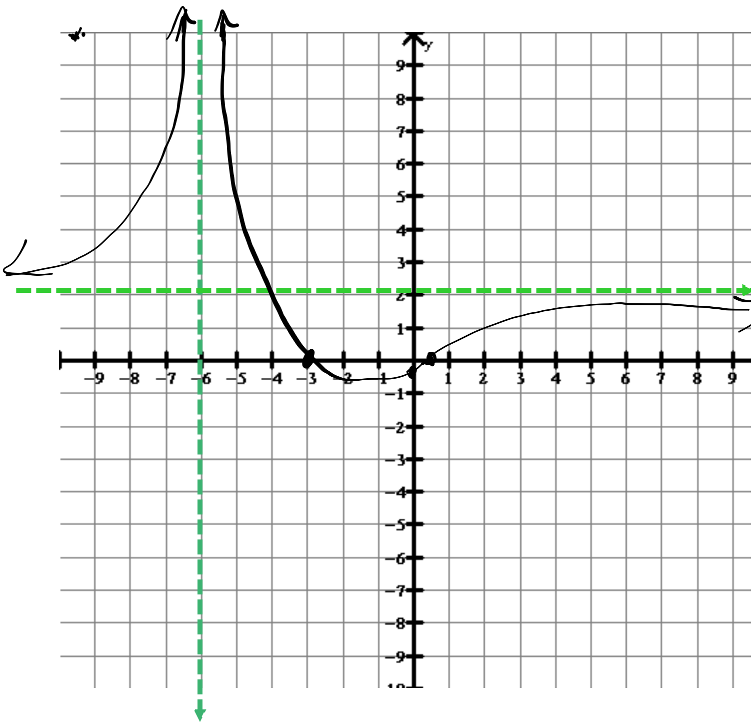
y- int $\frac{-3}{36} \sim -0.08$

VA $x=-6$

HA $y=2$

Beh.
$x=-6$

-6.1	-5.9
$\frac{-}{-}$	$\frac{-}{-}$
$+$	$+$
$+\infty$	$+\infty$



$f(x) = \dfrac{2x^2+5x-12}{x^2-x-2}$; $f(x) = \dfrac{(2x-3)(x+4)}{(x-2)(x+1)}$.

D: $x \neq 2, -1$
cannot ÷ by 0

x-int: $\frac{3}{2}$ and -4

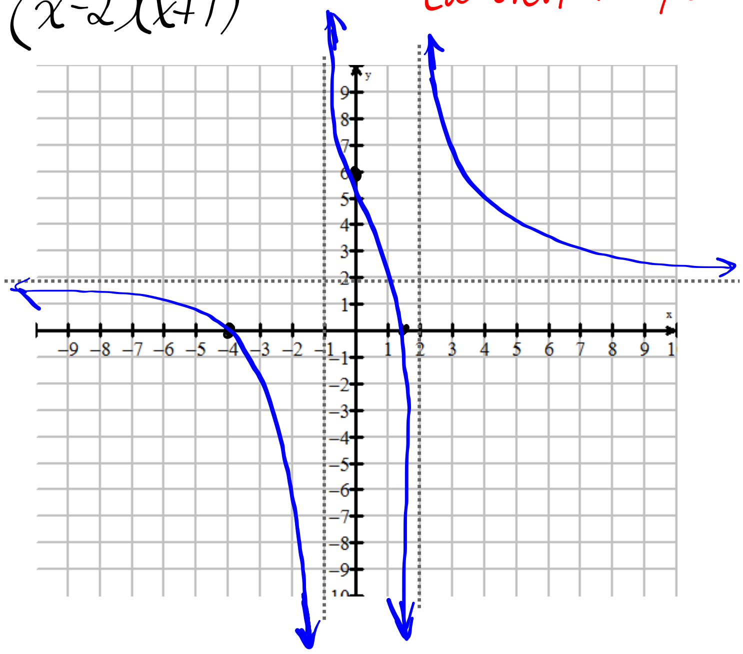
y-int: 6

VA: $x=2$, $x=-1$

HA $y = \frac{2}{1} = 2$ $y=2$

Behaviour at VA $f(x) = \dfrac{(2x-3)(x+4)}{(x-2)(x+1)}$

$x=-1$		$x=2$	
-1.1	-0.9	1.9	2.1
$-\ +$	$-\ +$	$+\ +$	$+\ +$
$-\ -$	$-\ +$	$-\ +$	$+\ +$
$-\infty$	$+\infty$	$-\infty$	$+\infty$



sketch: $y = \dfrac{x^2-8x-50}{25-4x^2}$; $y = \dfrac{(x^2-8x-50)}{(5-2x)(5+2x)}$

x-int: QF $\dfrac{8 \pm \sqrt{264}}{2}$ < 12.12, -4.12

y-int: -2

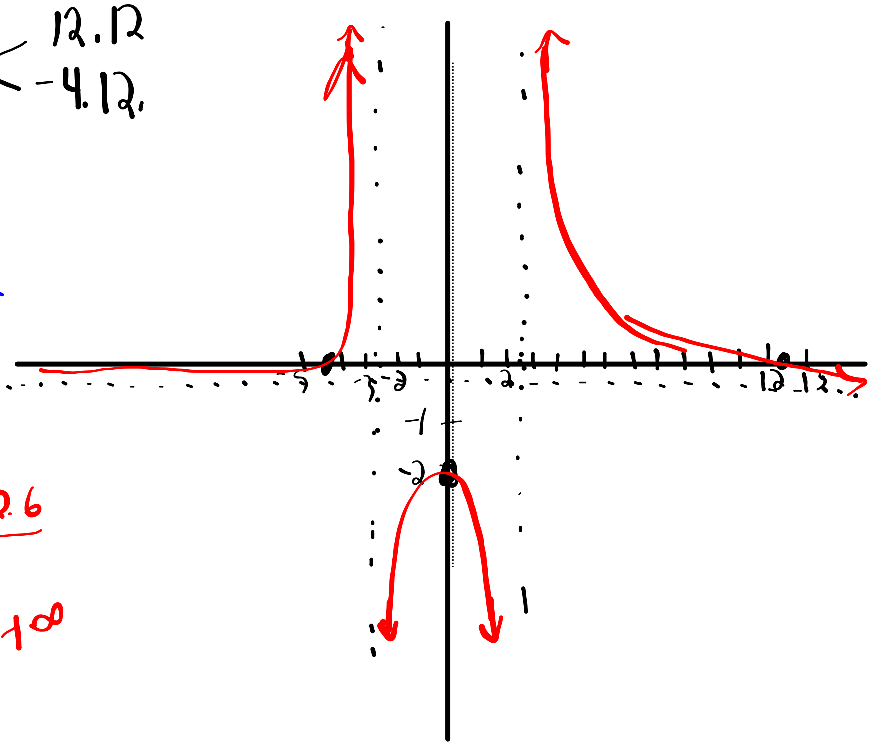
VA: $x = 5/2$, $-5/2$
2.5, -2.5

HA: $y = -1/4$

Beh:

-2.6	-2.4
$\frac{-}{+\,-}$	$\frac{-}{+\,+}$
+100	-50

2.4	2.6
$\frac{-}{+\,+}$	
$-\infty$	$+\infty$



Try:

$$y=\frac{3x}{x^2-2x-8}$$

Answer

$f(x) = \dfrac{3x}{x^2-2x-8}$ factor if possible $f(x) = \dfrac{3x}{(x-4)(x+2)}$

x-int: 0
y-int: 0
VA: x=4 , x=-2
HA: y=0/3 =0

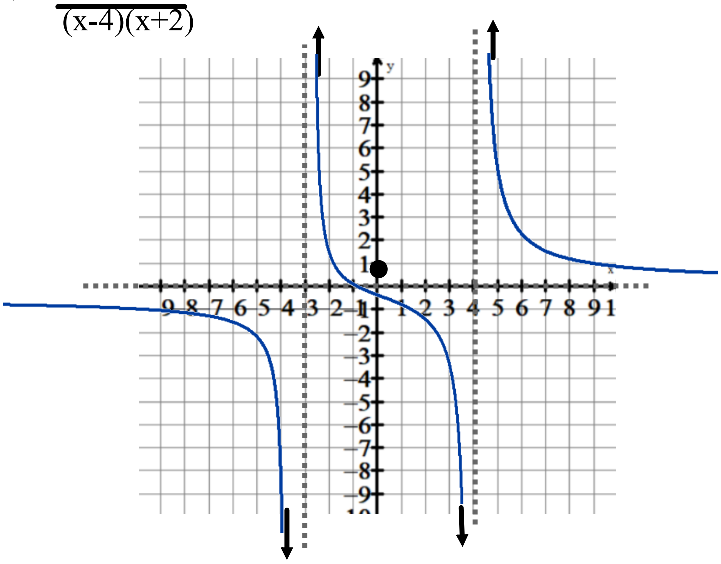
Behaviour $\dfrac{3x}{(x-4)(x+2)}$

x=-2

-2.1	-1.9
- -	-
-	- +
$-\infty$	$+\infty$

x=4

3.9	4.1
+	+
- +	+ +
$-\infty$	$+\infty$



Sketch $y=\dfrac{2x^2+5x-3}{x^2+2x-8}$

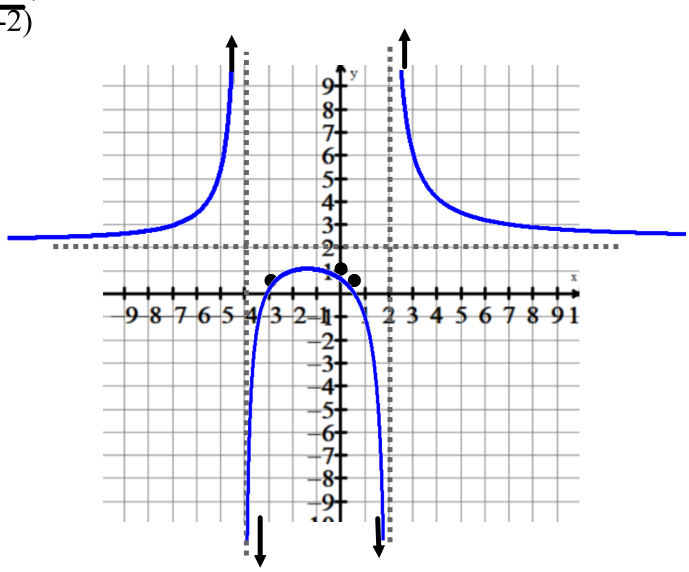
Answer

$y=\frac{2x^2+5x-3}{x^2+2x-8}$, $f(x) = \frac{(2x-1)(x+3)}{(x+4)(x-2)}$

x-int: 1/2 , -3
y-int: 3/8
VA: x= -4 , x= 2
HA: y= 2

Behaviour $\frac{(2x-1)(x+3)}{(x+4)(x-2)}$

x= -4		**x= 2**	
-4.1	-3.9	1.9	2.1
- -	- -	+ +	+ +
- -	+ -	+ -	+ +
$+\infty$	$-\infty$	$-\infty$	$+\infty$



Sketch: $$y = \frac{4x^2-3x-10}{x^2+x-2}$$

Answer

Sketch: $y = \dfrac{4x^2-3x-10}{x^2+x-2}$; $y = \dfrac{(4x+5)(x-2)}{(x+2)(x-1)}$

x-int: $-\frac{5}{4}$ & 2

y-int: 5

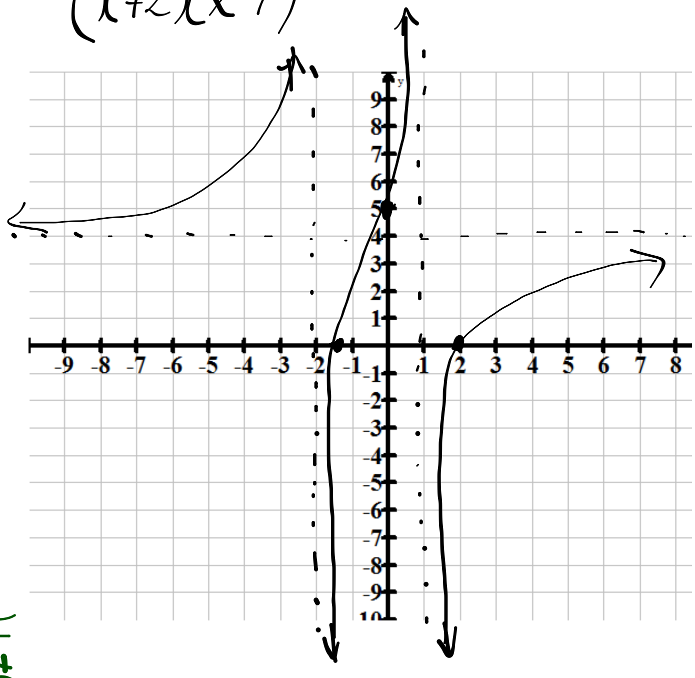
HA $y=4$

VA $x=-2$, $x=1$

Behaviour at VA:

Behav: at VA -2 , 1

-2.1	-1.9	0.9	1.1
$- \; -$	$- \; -$	$+ \; -$	$+ \; -$
$- \; -$	$+ \; -$	$+ \; -$	$+ \; +$
$+\infty$	$-\infty$	$+\infty$	$-\infty$



$$f(x) = \frac{x+3}{x+4}$$

Answer

$f(x) = \frac{x+3}{x+4}$

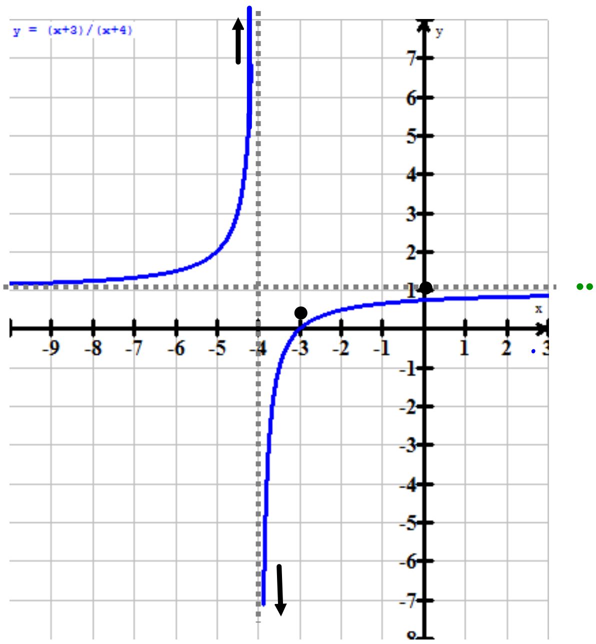
x-int: -3
y-int: 3/4

Vertical Asymptote x = -4

Hortizontal Asymptote y = 1

x = -4

-4.1	-3.9
$\frac{-}{-}$	$\frac{-}{+}$
$+\infty$	$-\infty$



Sketch

$y = x^2+x-9$

$(x-3)+9$

$$y = \frac{5x^2-9x-18}{x^2-6x+9}$$

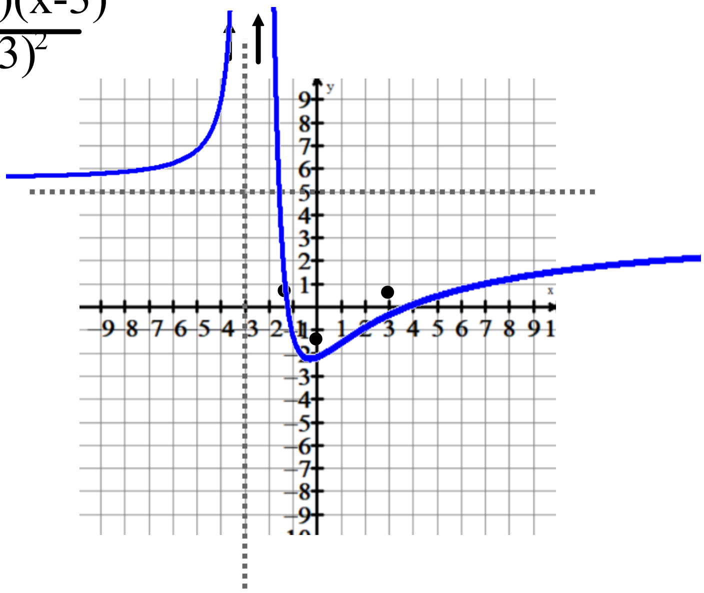
Answer

$$y = \frac{5x^2-9x-18}{x^2+6x+9} \qquad y = \frac{(5x+6)(x-3)}{(x+3)^2}$$

x-int: -6/5 , 3
y-int: -2
VA: x= -3
HA: y= 5

Behaviou $\frac{(5x+6)(x-3)}{(x+3)^2}$

-3.1	-2.9
- -	- -
+	+
$+\infty$	$+\infty$



Attachments

HA, VA, Intercepts.doc

graph sheet 1.doc